Convergence of an infinite series

Polytope

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In this note we will prove that

$$\sum_{k=1}^{\infty} \frac{\sin k}{k}$$

is convergent. We will use the sum-product identity

$$\cos(k - 1/2) - \cos(k + 1/2) = 2\sin(1/2)\sin(k)$$

which is easily verified by expanding the left side using the sum and difference identities for cosine.

Now,

$$\begin{aligned}
\sum_{k=1}^{n} \frac{\sin k}{k} &= \frac{1}{2\sin(1/2)} \sum_{k=1}^{n} \frac{\cos(k-1/2) - \cos(k+1/2)}{k} \\
&= \frac{1}{2\sin(1/2)} \left(\sum_{k=1}^{n} \frac{\cos(k-1/2)}{k} - \sum_{k=1}^{n} \frac{\cos(k+1/2)}{k} \right) \\
&= \frac{1}{2\sin(1/2)} \left(\sum_{k=0}^{n-1} \frac{\cos(k+1/2)}{k+1} - \sum_{k=1}^{n} \frac{\cos(k+1/2)}{k} \right) \\
&= \frac{1}{2\sin(1/2)} \left(\cos(1/2) + \sum_{k=1}^{n-1} \frac{\cos(k+1/2)}{k+1} - \sum_{k=1}^{n-1} \frac{\cos(k+1/2)}{k} - \frac{\cos(n+1/2)}{n} \right) \\
&= \frac{1}{2\sin(1/2)} \left(\cos(1/2) - \sum_{k=1}^{n-1} \frac{\cos(k+1/2)}{k^2+k} - \frac{\cos(n+1/2)}{n} \right).
\end{aligned}$$

It is easy to show that

$$\sum_{k=1}^{\infty} \frac{\cos(k+1/2)}{k^2+k}$$

is an absolutely convergent series, by comparing it to the convergent series

$$\sum_{k=1}^{\infty} \frac{1}{k^2+k} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right) = 1.$$

Moreover,

$$\lim_{n\to\infty} \frac{\cos(n+1/2)}{n} = 0.$$

Therefore,

$$\sum_{k=1}^{\infty} \frac{\sin(k)}{k} = \frac{1}{2\sin(1/2)} \left(\cos(1/2) - \sum_{k=1}^{\infty} \frac{\cos(k+1/2)}{k^2+k} \right).$$

In particular, the series converges.